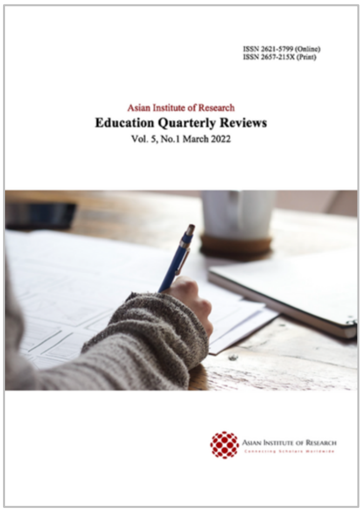

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The Interpretations of the Mathematics Teachers Taking Post-Graduate Education to the Expression "You can't add apples and pears"

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Abstract
The study aims to explore how the mathematics teachers taking post-graduate education use their Algebra knowledge for the expression, "You can't add apples and pears" that is expressed in almost every segment of the society, in creating mathematical content. The data of the research consisted of the written answers of the thirty-seven (37) teachers, who had taken the Algebra Teaching course, to the question "There is a rule in mathematics as everyone says; explain the statement 'You can't add apples and pears' mathematically" in the exams from the 2008-2009 academic year to, including 2015-2016 academic year. The participants took the algebra education in the courses named Algebra and Introduction to Algebra courses and sufficient algebra education was provided to answer the question in the courses. The document analysis method was employed in the study and the data were analyzed with the descriptive content analysis. The results indicated that most of the teachers were aware of their algebra knowledge, the number and quality of the mathematical contents in the answers were at a satisfactory level, but the visuals used in some contents were not appropriate.

Keywords: Algebra Knowledge, Algebra Teaching, Awareness, Math Teachers, Fruit-Vegetable Algebra

1. Introduction

1.1 Introduce the Problem

Individuals use the "You can't add apples and pears" expression to state that a work or operation is not correct. Even, with "There is a rule in mathematics. You can't add apples and pears" and by adding the "mathematics" word to the content, they imply that the verification is provided with mathematics and no further proof is needed. Besides, as is can be seen from the examples below, the expression "You can't add apples and pears" is used in many different areas such as design and medicine.

Example 1. "… It is the first thing that our teachers teach us when we start the primary school, perhaps it is one of the most concrete things that we learn in mathematics. You can't add apples and pears!"

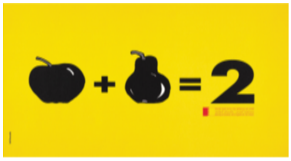

Image 1: Sadık Karamustafa, “Sharing Jerusalem: Two Capitals for Two States” Graphist 7th International Graphic Design Days, 2003.

Nevertheless, things have changed in this work of the designer. When it comes to the limitless of the design, it has been tried to emphasize that apples and pears can be added and the result will be two, that is, two different societies can exist together…” (Okur, 2014).

Example 2. “… Reductive expressions are incorrect. Reductive expressions demonstrate apples and pears equal: both are fruits; they both grow on trees, both etc. etc. Hermeneutics teaches us to avoid definitions. Because definitions are reductive…” (Uzun, 2014).

Example 3. “… Children are told that “you can’t add apples and pears” when teaching mathematics. As everyone knows that apples and pears cannot be added, they also know that religious issues cannot be approached with positive science methods…”(Bilgili, 2014).

Example 4. “…One of the most heard statements in mathematics is the expression “you can’t add apples and pears”. Is it possible to add apples and pears? The answer of this question can be conditionally “yes”…” (Akdemir, 2009).

Example 5. “…words can be added with words, sentences with sentences, words cannot be added with sentences. Isn’t this the golden rule of mathematics? “you can’t add apples and pears”. When I first heard this, I was in primary school. I remember I thought “Why can’t they be added? Two apples plus three pears equal five. It equals five, of course, but five what? I wasn’t aware that I should add fruits, two things of the same kind, not apples and pears…” (Hepçilingirler, 2004).

Example 6. “…The “point” unit is probably the most used unit after the “US Dollar”. Because we measure intelligent with points as well as anxiety… Exam mark is also a point and KPSS score, too… Even those who claim that apples and pears cannot be added, do not mind adding the interview score and the score obtained from ALES exam…” (Bilgili, 2014).

Example 7. “…Things that are different from each other by a certain extent are also logically contradictory, as they will not be the same thing. Eventually, it is like to compare apples with pears (one unit with five units)…” (Gümüş, 2005).

Example 8. “…the average amount of antibiotics used for patients will be given or the average amount of antibiotics used for 1000 people in the community in a day (in a way, it becomes possible to add apples and pears)…” (Töreci, 2003).

Example 9. “…Who will win can be predicted when different animals compete in the same running race. In this addition operation, it is similar with the expression that teachers use frequently: you can’t add apples and pears. The fact that the lion and the tortoise are in the same race and that they will race on the same track can show the equality, but it is obvious that it is not fair in terms of physical features…”(Kuzucu, 2018).

It is known that the mathematical symbols and terms are used except for their mathematical meanings and this is most common with the "+" symbol and the "addition" operation (Çetin et al., 2013: Delice & Sür 2015). In addition, as it has been stressed in the examples above, the expression “You can’t add apples and pears” is used in schools and by teachers. It is argued that an approach named “fruit-vegetable algebra” is used by teachers in teaching the algebraic expressions subject by the teachers (Booth, 1988; Eroğlu, 2016; Kuchemann 1978; Rosnick, 1981; Tirosh, Even & Robinson, 1998; Yıldız, 2020), that this approach may cause mistakes and misconceptions (Eroğlu, 2016; Pimm, 1987; Tirosh, Even & Robinson, 1998). Teachers should consider that while this approach makes it easier for students to learn by concretizing the concept, meets the needs that arise at that moment, at the same time, this approach influences that students' general perceptions that they develop related to mathematics (Eroğlu, 2016) and that it is used independent from the textbooks (Ubuz & Sarpkaya, 2014).

1.2 Explore Importance of the Problem

It is claimed that a teacher, who has a deep content knowledge (Ball, 1990; Fennema, Sowder and Carpenter, 1999), will also plan and implement the teaching process effectively (Ball, Thames & Phelps, 2008; Huang & Kulm, 2012; Wasserman, 2016). The mathematics teachers taking the post-graduate education have the algebra content knowledge as they had taken the algebra course during their post-graduate education. How the teachers, who take Algebra Teaching course that takes place in the postgraduate curriculum, use their knowledge to create mathematical content for the expression "You can't add apples and pears" that almost every segment of the society use was considered to be worth searching.

1.3. Research Purpose

We aimed to explore how the post-graduate mathematics teachers taking post-graduate education use their Algebra knowledge for the expression, "You can't add apples and pears" that is expressed in almost every segment of the society, in creating mathematical content.

1.4. Research Problem

Can the mathematics teachers taking the post-graduate education use their algebra knowledge to explain the expression "You can't add apples and pears"? In other words, are they aware of their knowledge? This research problem was interpreted with the sub-problems given below.

1. Which answers did the mathematics teachers taking post-graduate education give for the expression "You can't add apples and pears"?
2. What is the mathematical content in the answers for the "You can't add apples and pears" expression?

2. Method

In this study, which aims to explore the awareness of algebra knowledge of the mathematics teachers taking the postgraduate education, the document analysis method was employed since we wanted to analyse the data obtained form the exams, in which the question "As everyone says, there is a rule in mathematics: You can't add apples and pears" was asked. The document analysis is the investigation of the written materials consisting of the information about the problem or problems that are wanted to be explored. The document analysis can also be applied alone to make sense of the written materials (Yıldırım & Şimşek. 2016).

2.1. Research Universe and Sample

The data of the research consisted of the written answers of the thirty-seven (37) math teachers. The participants took the algebra education in the courses named Algebra and Introduction to Algebra courses and sufficient algebra education was provided to answer the question in the courses.

2.2. Data Collection Tools

As the data collection tool in the research, the exam papers consisting of the question "Please state the mathematical explanation of 'as everyone says, there is a rule in mathematics: You can't add apples and pears'" asked as a question in the Algebra Teaching course given in the post-graduate program with thesis in each academic year from the 2008-2009 to 2015-2016 academic year including the last year were applied. Since the exam was given as a research problem, one week of duration was given to the teachers to answer the question. There were answers between one page to seven pages.

2.3. Data Collection Process

The data were obtained from the written answers to the examination questions of the Algebra Teaching course given in the post-graduate program with thesis in each academic year from the 2008-2009 to 2015-2016 academic year including the last year. That is, the study is a longitudinal study that consists of 8 (eight) years.

2.*4. Data Analysis*

To analyse the data obtained within the scope of the research, the descriptive analysis method, which aims to present the obtained findings to the reader in a way as classified and interpreted, was implemented in four stages (Yıldırım & Şimşek 2016) as;

1. Creating themes and sub-themes,
2. Processing the data according to the themes and sub-themes,
3. Findings,
4. Interpreting the findings.

Creating themes and sub-themes. With the help of the data obtained by reviewing the documents, the themes and sub-themes of these themes have been created and are presented in Table 1 below

Table 1: Themes and sub-themes of these themes of the data obtained from the documents.

Themes	Sub-themes
Can't be added	Since the total does not specify a certain result Addition is a binary operation. The elements of the binary operation must belong to the same set. Since Apples and Pears are not the same genus/unit/variable, they can't be added. To add the two sets, the domain set and codomain set must be equal. Can't be added as they don't have consecutive.
It depends on the situation	If the unit is regarded as fruit, it can be added, otherwise not. It can be added if the studied universe/cluster is taken as fruit. If a universe/cluster is taken separately, such as apples and pears, the addition cannot be performed since the closure feature is not provided.

Data processing by themes and sub-themes. The data obtained from the exams were transferred to the computer environment by coding one by one and were classified according to the themes and sub-themes in Table 1.

3. Results

3.1 Findings

3.1.1. The findings related to the 1st sub-problem as Which answers did the mathematics teachers taking post-graduate education give for the expression “You can’t add apples and pears”? are presented and interpreted below by using direct citations.

The distribution of the answers by the mathematics teachers taking post-graduate education to the question “you can’t add apples and pears” by the themes and sub-themes of these themes are presented in Table 2 below as frequency (f) and rate (%).

Table 2: The distribution of the answers by the mathematics teachers taking post-graduate education to the question "you can't add apples and pears" by the themes and sub-themes of these themes.

What is the mathematical meaning of the expression "you can't add apples and pears"?			
	Themes	f	%
	Can't be added	28	76
	Depends on the situation	9	24
	Total	37	100
Themes	Sub-themes		
Can't be added	Since the total does not specify a certain result	8	29
	Addition is a binary operation. The elements of the binary operation must belong to the same set.	6	21
	Since Apples and Pears are not the same genus/unit/variable, they can't be added.	6	21
	To add the two sets, the domain set and codomain set must be equal.	5	18
	Can't be added as they don't have consecutives.	3	11
	Total	28	100
It depends on the situation	If the unit is regarded as fruit, it can be added, otherwise not.	6	67
	It can be added if the studied universe/cluster is taken as fruit. If a universe/cluster is taken separately, such as apples and pears, the addition cannot be performed since the closure feature is not provided.	3	33
	Total	9	100

As it can be seen in Table 2, while %76 of the teachers (28 teachers) made a comment as "can't be added", %24 (9 teachers) claimed as "it depends on the situation". Since %29 of the teachers (8 teachers), who claimed that "it can't be added", stated that "Since the total does not specify a certain result"; %21 (6 teachers) that "Addition is a binary operation. The elements of the binary operation must belong to the same set."; in addition, %21 (6 teachers) claimed that "Since Apples and Pears are not the same genus/unit/variable, they can't be added"; %18 (5 teachers) that "To add the two sets, the domain set and codomain set must be equal" and %11 (3 teachers) that "Can't be added as they don't have consecutives."

There are some examples among the teachers' answers for each sub-themes of the "can't be added" theme below. The answer of S21 coded teacher among the teachers in the "Since the total does not specify a certain result" sub-theme is presented below.
The answer of the S21 coded teacher.
$(a,b), (c,d) \in N \times N$
Let's show how to add 5 apples and 3 apples.
Let's take Apple=(a,b)
$5(a,b)+3(a,b) = (5a,5b)+(3a,3b)$
$= (8a,8b)$
$= 8(a,b)$
=8 apples
Taking Apple=(a,b) and Pear=(c,d);
$(a,b)+(c,d)=(a+b,c+d)$
Since the total does not specify a certain result, apples and pears can't be added.
The S1 coded teacher's answer in the "Addition is a binary operation. The elements of the binary operation must belong to the same set" sub-theme.
The answer of S1 coded teacher.
Two operations should have the same algebraic form. The elements of binary operation must belong to the same set. Addition is a binary operation. There must be a function as $XxX \rightarrow X$. If $X \neq \Phi$ and $Y \neq \Phi$ and X, Y are different,

$X \times Y \rightarrow X$ can't be a binary operation. So, as X={x; x∈apples}, Y={ y; y∈Pears}, should the function go to X or Y so that it's a binary operation? Here, it is not certain that the function will go to X or Y sets.
Here, let A= {a; a∈all fruits}. It can be $A \times A \rightarrow A$. However, here;
If, Apple + Pear= Banana, the addition of any fruit with Banana is the Banana itself. This is a binary operation. But, it is not well-defined. Because, we have assumed that the addition of Apple and Pear is Banana. Someone else may assume as Strawberry or Watermelon. In order for the binary operation to be well-defined, a set, that goes back to itself from the Cartesian product of itself, must be a well-defined function.
The S34 coded teacher's answer to "Since Apples and Pears are not the same genus/unit/variable, they can't be added" sub-dimension.
The answer of S34 coded teacher
Let $a,b \in N$ be $a+b=c$ and $c \in N$, (The closure property of the set of natural numbers according to the addition operation).

$$a+b=c$$
$$a.1+b.1=c.1 \rightarrow (I)$$
$$1.(a+b)=1.c \rightarrow (II)$$

The natural number "1", which is used in the operations above, represents the same object every time. Otherwise, it will not be possible to put the natural numbers a and b in the "1" common bracket, which is made in the transition from the Ist situation to the IInd situation. In this case, "(c).1" cannot be written as $c \in N$ on the right side of the equation. Thus, the result that there is no $c \in N$ with $a,b \in N$ such that $a+b=c$ emerges. As this result contradicts the closure property of the natural numbers according to the addition operation, the necessity that all the 1's written in I operation to represent the same object emerges.
The answer of S36 coded teacher "To add the two sets, the domain set and codomain set must be equal." Sub-dimension is presented below.
The answer of S36 coded teacher
What is set: A well-defined collection of objects is called a set.
Example. Natural numbers set, integers set, etc.
Apples set: The set that consists of apples are the apples set. Since the set is well-defined object, it is necessary to define the apple set according to its properties.
Features of apples:

- It can grow in all four seasons
- It can be in red and yellow colour.
- Its seed is like a spherical structure.
- The leaves of the tree are in the shape of ellipse. etc.

Pears set: The set that consists of pears are the pears set. Since the set is well-defined objects, it is necessary to define the pears set according to its properties.
Features of pears:

- It can grow in summer.
- It has a yellowish colour.
- Its seed is spicular.
- Its leaves are in narrow shape. etc.

Now, let's see whether apples and pears form the same set or not:
Theorem. Apple and pear are not in the same set.
Proof. Let apple and pear be in the same set, then, each apple in the apples set (E) should be even in the pears set (A).
That is;

$$0 \in E \Rightarrow 0 \in A, \forall\, n \in E \Rightarrow \forall\, n \in A$$

Since 0 (zero) is an empty set, it is a subset of every set;

$$0 \in E \Rightarrow 0 \in A$$

Now, let's take any n element from the E (apples set) set. This element must provide the element characteristics above. It one of these characteristics does not provide their characteristics of the A (pears set) set, we will contradict with our acceptance.

The elements of the E set can group in all seasons. The element of A set grows in summer. In this case, our acceptance is wrong. That is;

each $n \in E \Rightarrow$ each $n \notin A$

in this case, E set is different from A set.

Addition: Let $m \in N$ be a set. In this situation;

T1) $t_m(0)=m$

T2) For each $n \in N$ the $t_m(n^+) = (t_m(n))^+$ has only one $t_m: N \to N$ function that meets the conditions.

It must be $t_m(0)=m$. If $m \in E$, it can't be $m \notin A$.

$t_m: N \to N$ define function is $m \in E$ it can't be $m \in A$.

That a function is defined to $N \to N$, is expected that the elements of E set and A set to be the elements of N set. It is only possible to define a set that consists the E and A sets. Otherwise, it is not possible to mention such a situation. The reason why it can't be mentioned is our way out.

The definition above indicates that: in order to perform addition operation on a set, it must be defined on the same set. The apples set and pears set are different sets. This case indicates us that apples and pears cannot be added.

The answer of S26 coded teacher in the "Can't be added as they don't have consecutives" sub-dimension is presented below

The answer of the S26 coded teacher

First, let's define the addition operation on consecutive sets and natural numbers.

Consecutive set: If $\emptyset \in X$ and each $A \in X$ is $A^+ \in X$, X is the consecutive set.

Addition: Let the natural numbers m, $n \in N$ be given. In this case,

t_1: $t_m(0)=m$

t_2: there is one function $t_m: N \to N$ that provides the $\forall n \in N$ and $t_m(n^+) = (t_m(n))^+$ conditions. This function is called as the addition function. For instance, let's add 2 and 3.

$t_2(0) = 2$

$t_2(1) = t_2(0^+) = (t_2(0))^+=3$

$t_2(2) = t_2(1^+) = (t_2(1))^+=4$

$t_2(3) = t_2(2^+) = (t_2(2))^+=5$ is found.

That is expected from us is to add 3 apples and 4 pears. Such an operation cannot be performed. Because, addition is the operation of counting on. That is, addition operation is a consecutive operation. While adding 3 and 4, we get the consecutive of 3 4 times. If the definition of consecutive set is considered, it should be $a^+ \in X$ for $a \in X$. However, the consecutive of the 3 that we take from the apples set is 4 apples in the apples set. So, the addition operation cannot be performed between two different set elements since we can't get the consecutive of the number, we get from one of the two different sets from the other set. A new operation should be defined to perform this operation.

There are examples of the teachers' answers for the two sub-themes of the "it depends on the situation" theme below.

The answer of the S7 coded teacher that takes place in the "If the unit is regarded as fruit, addition operation can be performed, otherwise not" theme is presented below.

The answer of S7 coded teacher

The expression that apples and pears cannot be added is not correct. Because, it can be added according to the universe that we study. If we accept the universe of apples and pears as fruit, we can gather them together under the fruit set.

For example;

3 apples + 4 pears = 7 fruit

However, if we perform the addition operation by accepting the apples as a different universe and pears as a different universe, among the characteristics of being a group of addition operations, the closure feature is not provided. So, we can't perform the addition operation.

The answer of the S13 coded teacher that takes place in the "If a universe/set is taken separately, such as apples and pears, the addition cannot be performed since the closure feature is not provided" sub-theme is presented below

The answer of S13 coded teacher

An apple + a pear can't be added.
An apple +a pear =2 fruits. When written so, addition operation is performed. Because, both are gathered under a common set (fruit). Let's compare these two questions.

- How many pears should we add to the basket with 2 apples, so there will be 8 apples in the basket?
- How many pears should we add to the basket with 2 apples, so there will be 8 fruits in the basket?

As it is seen in the first question, if apples and pears were collected, we would find the desired number of apples when we add pears to the basket. We can add apples and pears in the second questions, because we name both of them in the common set (fruit).

Let's give a different example.

5 hens +3 cocks cannot be added.

We can state 5 hens+3cocks = 8 animals. Because, we can state a common set (animal).

3.1.2. The Findings related to the second sub-problem as What is the mathematical content in the answers for the "You can't add apples and pears" expression?
The findings related to the answers of the mathematics teachers taking the post-graduate education to the question "you can't add apples and pears" are presented in Table 3 below.

Table 3: The distribution of the algebraic content and findings of the answers of mathematics teachers taking the post-graduate education to the question "you can't add apples and pears" by the themes and sub-themes of these themes.

Themes	There is an algebraic content in answers		There is no an algebraic content in answers		Total	
	f	%	f	%	f	%
Can't be added	23	82	5	18	28	100
It depends on the situation	2	22	7	78	9	100
Total	25	68	12	32	37	100

As it is seen in Table 3, while there is an algebraic content in the answers of the %82 of the teachers (23 teachers) in the theme of " can't be added", there is an algebraic content in the answers of the %22 of the teachers (2 teachers) in the "it depends on the situation" sub-theme. In addition, while there is no algebraic content in the answers of the %28 of the teachers (5 teachers) in the sub-theme of "can't be added", there is no algebraic content in the answers of the %78 of the teachers (7 teachers) in the "it depends on the situation" sub-theme.

While some mathematical contents indicate how strongly the teachers form their algebra knowledge, some mathematical contents are such as an example of not being able to structure information.

The mathematical contents in the teachers' answers are presented in Table 4 below.

Mathematical Contents in the Answers		f	%
1	Binary operation	15	36
2	Function	6	14
3	Formation of natural numbers with Peona axioms	1	2
4	Formation of natural numbers using sets	1	2
5	Consecutively in sets and natural numbers	4	10
6	The binary addition operation in natural numbers	3	8
7	Addition in natural numbers with a constant number	5	12
8	Addition operation on exponential, radical, rational and irrational numbers	1	2
9	Group structure and characteristics	1	2
10	Indirect proof method	2	5
11	With reference use	3	7
Total		42	100

As it is seen in Table 4, the number of algebraic mathematical content that the teachers used in their answers are 42; this number is more than 25 which is the number of answers consisting of the mathematical expression given in Table 3. In addition, as it is understood from Table 4, the “binary operation” was used in the answers most (%36). 3 teachers presented the mathematical contents not directly but in an article content by using the reference in their answers.

While some mathematical contents indicate that how strongly teachers form their algebra knowledge, some mathematical contents, although they are few, are such as to be an example of not being able to structure knowledge. Some of the contents that will be an example of not being able to structure information are presented below.

The expression “… It is not possible to add the sets whose definition sets are different…” in the S23 coded participant’s answer is an example to confusing operations between sets and binary operations on sets.

The expression “…Positive integers, which form the basis of mathematics, were characterized axiomatically by the Italian mathematician Peano (1858-1932) in 1889 as follows…” in the S12 coded participant’s answer is an example of regarding Natural numbers defined very differently from each other and Positive Integers the same.
The visual in the answer of the S24 coded participant and presented in the Figure 1 below.

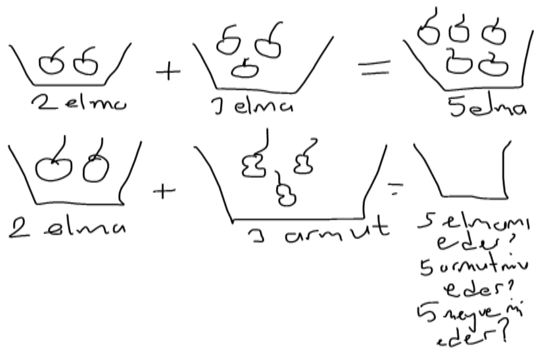

Figure 1: The visual in the answer of the S24 coded teacher

The visual in the answer of the S27 coded participant and presented in the Figure 2 below.

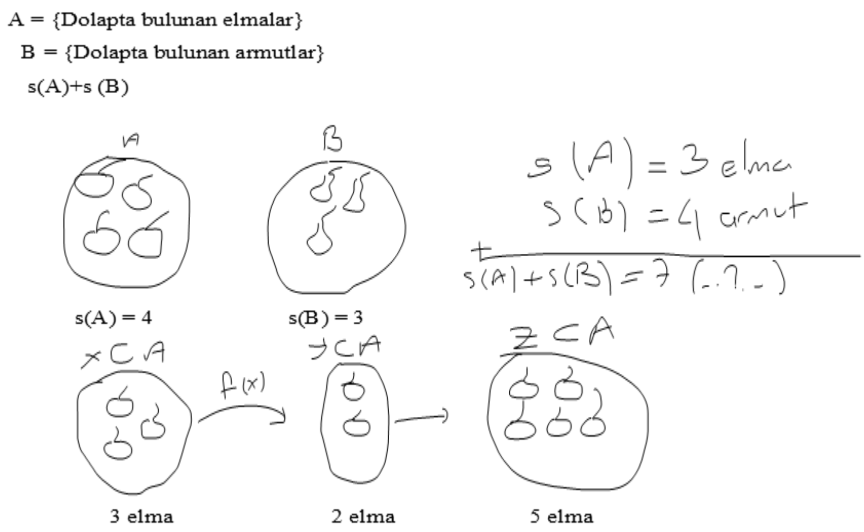

Figure 2: The visual in the answer of the S27 coded participant

The visuals in the Figure 1 and Figure 2 are used by the educators to visualize the addition operation, can cause difficulties in forming the concepts of set, operations in sets and binary operations in terms of mathematical content.

The examples indicating how strongly some teachers form their algebra knowledge are presented below.
The expression in the answer by the S36 coded participant
"…What is set: A well-defined collection of objects is called a set.
Example: Natural numbers set, integers set etc.
Apples set: The set of apples is called as apples set. Since the set is well-defined objects, it is necessary to define the apples set according to its properties. Properties of apples:

- It can grow in all four seasons
- It can be in red and yellow color.
- Its seed is like a spherical structure.
- The leaves of the tree are in the shape of ellipse. etc.

Pears set: The set that consists of pears is the pears set. Since the set is well-defined objects, it is necessary to define the pears set according to its properties.
Features of pears:

- It can grow in summer.
- It has a yellowish color.
- Its seed is specular.
- Its leaves are in narrow shape. etc.

Now, lets find if apples and pears create the same set:
Theorem. Apples and pears are not in the same set.
Proof. Let the apples and pears be the same set. Then, every element in the apples set (E) must be in the pears set (A), too. That is;

$$0\in E \Rightarrow 0\in A, \text{ each } n\in E \Rightarrow \text{each } n\in A$$

Since 0 (zero) is the empty set, it is the subset of each set. That is;

$$0\in E \Rightarrow 0\in A$$

Now, lets take an n element from any elements in the E (apples set) set. This element must provide the properties of the element above.
In the case, when one of these properties does not provide the properties of A (pears set) set, we contradict with our acceptance.
The elements of the E set grow in all four seasons. The elements of A set grow only in summer. In this case, our acceptance is incorrect. That is;

$$\text{each } n\in E \Rightarrow \text{each } n\notin A$$

in this case, the expression "…the E set is different from A set…" indicates that the teacher has formed the knowledge of algebra in a way that can carry it to current life.
The expression in the answer of the S35 coded participant to the question "…we should not confuse the addition operation with natural numbers with the union operation on sets…" indicates that the teacher forms his/her algebra knowledge at a level that can identify misconceptions.

4. Discussion

4.1. The results related to the 1st sub-problem as Which answers did the mathematics teachers taking post-graduate education give for the expression "You can't add apples and pears"?

When the findings related to the first sub-dimension were analyzed, while most of the teachers claimed that apples and pears can't be added, approximately one fourth of them stated that they can be added according to the situation of the apples and pears. This result indicates similarity with the results reached by Eroğlu, 2016 suggesting that " It was observed that the teachers used 1) adding similar terms, 2) fruit salad approach in the courses" and Yıldız, 2020, that " the teachers used the 'fruit-salad', 'similar terms' and 'non-similar terms' approaches during the algebraic expressions operation process."

The reasons of the teachers who answered as ‘cannot be added’ are gathered under the sub-themes “as the sum does not refer a result” with 8 (eight) teachers at most; “cannot be added as they don’t have consecutives” with 3 (three) teachers at least. These two reasons and the other reasons indicate that teachers associate the content of the question with the subjects of binary operation, formation of natural numbers and addition operation in natural numbers.

While the reasons of the teachers who claimed that ‘it depends on the situation’ were gathered under the sub-theme of “If the unit is taken as the fruit, addition operation can be performed, otherwise not” with 6 (six) teachers, 3 (three) teachers’ reasons were gathered under the sub-theme of “If a universe/set is taken as fruit, such as apples and pears, addition operation cannot be performed since the closure feature is not provided”. As it can be realized in these two reasons, the contents that cause the “can’t be added” judgement and the contents under the theme of can’t be added are almost the same. However, the content “If the unit is taken as the fruit, addition operation can be performed” and “If the universe/set is taken as fruit, addition operation can be performed” present a different point of view and is significant. If these discourses were supported by mathematical content, it would be an algebraically desirable solution.

4.2. The results related to the second sub-problem as What is the mathematical content in the answers for the “You can’t add apples and pears” expression?

There is a mathematical content in the answer’s majority of all the teachers (68%). While there was a mathematical content in the answers of the majority of the teachers (82%) under the theme of can’t be added, in the answers of only 2 (two) teachers, among the 9 (nine) teachers under the theme of it depends on the situation, there is a mathematical content. When the mathematical contents used in the answers are taken into consideration, it is seen that the teachers have sufficient algebra knowledge and are aware of their knowledge. This result contradicts with the result by Yıldız, 2020 suggesting that “The content knowledge of the teachers about the process of processing with algebraic expressions is not at a sufficient level.” This case may stem from the the Algebra Teaching course they took in the post-graduate degree in addition to the Algebra course taken by the teachers participating in this study at the undergraduate level. The result reached in the study by Eroğlu, 2016 suggesting that “with professional development activities, teachers show a great change and development in the knowledge and skills they use in their teaching, and this change and development takes place in the context of the use of representations and mathematics teaching knowledge” can be given as the reason for this.

The teachers used 11 (eleven) different mathematical contents in their answers. The mathematical contents that were used most were “binary operation” with 15 (fifteen) students and “function” with 6 (six) students. The mathematical contents that were used least were “Formation of natural numbers with Peona axioms”, “Formation of natural numbers using sets”, “Addition operation on exponential, radical, rational and irrational numbers”, “and “Group structure and characteristics” with 1 (one) teacher. This situation indicates that the teachers who refer mathematical contents in their answers have a deep content knowledge. This result overlaps with the result by Yıldız, 2020 suggesting that “They did not use alternative approaches to the mathematical meaning of the algebraic expression. For this reason, teachers should have a deep content knowledge about the structural feature of algebraic expression, which includes the mathematical meaning.”

The visuals that are used in the Figure 1 and Figure 2 and mathematical symbols are regarded as perceptual rather than a mathematical basis by teachers. Since such uses may lead difficulties in teaching the real meanings of mathematical symbols, it would be more appropriate not to be applied by mathematics teachers at least.

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