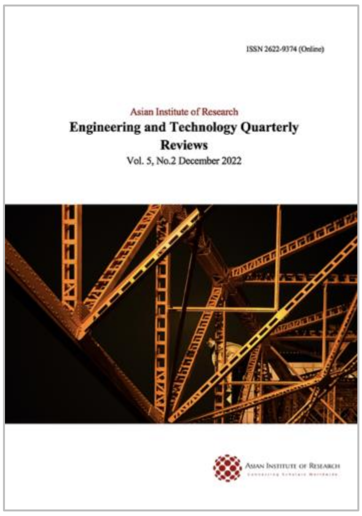

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On Operators Preserves in Normed Inner Product Spaces

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Abstract
We consider that a finite dimensional real normed linear space X is an inner product space if for any linear operator T on X, T preserving its norm at $e_1, e_2 \in S_X$ implies T attains its norm at span$\{e_1, e_2\} \cap S_X$. We prove by the convexity theorem.

Keywords: Convex, Operators, Orthogonality, Norm Space, Linear Operators, Inner Product Space

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space. Let $B_X = \{x \in X: \| x \| \leq 1\}$ and $S_X = \{x \in X: \| x \| = 1\}$ be the unit ball and the unit sphere of the normed space X respectively. Let $B(X,Y)(K(X,Y))$ denote the set of all bounded (compact) linear operators from X to another real normed space Y. We write $B(X,Y) = B(X)$ and $K(X,Y) = K(X)$ if $X = Y$. $T \in B(X,Y)$ is said to attain its norm at $x_0 \in S_X$ if $\| Tx_0 \| = \| T \|$. Let M_T denote the set of all unit vectors in S_X at which T attains [Watson, 1992] norm, i.e.,

$$M_T = \{x \in S_X: \| Tx \| = \| T \|\}$$

Let X be a normed space over the field $K \in (R, C)$; then for $x, y \in X$. We call the relation $\perp_B$, a Birkhoff [Li and Schneider, 2002],[Bhatia and Šemrl, 1999], [Chmielowski and Wójcik, 2010] ,[NSKI and AZYCH, 2005] orthogonality (often called a Birkhoff-James orthogonality). There is no unique way how to transfer the notion of orthogonality from inner product spaces to normed spaces. Perhaps the most useful is the notion of Birkhoff orthogonality; however many other can be used. One can also consider an axiomatic definition of the orthogonality relation and the orthogonality space. The notion of Birkhoff–James in [Lumer, 1961], [Alonso et al., 2012] orthogonality plays a very important role in the geometry of Banach spaces. For any two elements $x, y \in X$, x is said to be orthogonal to y in the sense of Birkhoff–James, written as $x \perp_B y$, if and only if

$$\| x \| \leq \| x + \lambda y \| \ \forall \lambda \in \mathbb{R}.$$

Similarly, [Paul et al., 2016a], [Paul et al., 2016], [Turnšek, 2005], [Sain, 2017], and [Alonso et al., 2012] define for $T, A \in B(X, Y)$, T is said to be orthogonal to A, if and only if

$$\| T \| \leq \| T + \lambda A \| \ \forall \lambda \in \mathbb{R}.$$

C. Alsina, J. Sikorska, M. S. Toms,[Carlsson, 1962] define. Let X be a linear space over $\mathbb{K}(= \mathbb{R} \quad or \quad \mathbb{C})$. The inner product (scalar product) is a function

$$\langle \cdot, \cdot \rangle : X \times X \longmapsto \mathbb{K}$$

such that:

1. $\langle x, x \rangle \geq 0$
2. $\langle x, x \rangle = 0$ if and only if $x = 0$
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$;
4. $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$;
5. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

for all $x, y_1, y_2 \in X$ and all $\alpha \in \mathbb{K}$ $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$,
$\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$,
for all $x, y_1, y_2 \in X$ and $\alpha \in \mathbb{K}$
For a space with an inner product we define

$$\| x \| = \sqrt{\langle x, x \rangle}.$$

C.Alsina, J.Sikorska,M.S.Toms, In [Alsina et al., 2010] define. A pair $(X, \|\cdot\|)$ is called a real normed linear space provided that X is a vector space over the field of real numbers $\mathbb{R}$ and the function $\|\cdot\|$ from X into $\mathbb{R}$ satisfies the properties:

1. $\| x \| \geq 0$ for all x in X,
2. $\| x \| = 0$ if and only if $x = 0$,
3. $\| \alpha x \| = |\alpha| \| x \|$ for all x in X and α in $\mathbb{R}$,
4. $\| x + y \| \leq \| x \| + \| y \|$ for all x and y in X.

The function $\|\cdot\|$ is called a norm and the real number $\| x \|$ is said to be the norm of x. In the real line $\mathbb{R}$ the only norms are those of the form $\| x \| = |x|, x \in X$, where $| \cdot |$ denotes the absolute value $|x| := max(x, -x), x \in \mathbb{R}$. In general, for all x, y in X we have
$| \| x \| - \| y \| | \leq \| x - y \| \leq \| x \| + \| y \|$,

In [Alsina et al., 2010] and [Birkhoff, 1935] so introducing the mapping d from $X \times X$ into $\mathbb{R}$ by
$d(x, y) := \| x - y \|$,

for all x, y in X, we infer that d is a metric induced by the norm $\|\cdot\|$, so (X, d) is a metric space and therefore a topological space. With respect to the metric topology, the norm $\|\cdot\|$ is continuous and the topology induced by the norm is compatible with the vector space operations, i.e., $\mathbb{R} \times X \ni (\alpha, x) \longmapsto \alpha x \in X$ and $X \times X \ni (x, y) \longmapsto x + y \in X$ are continuous in both variables together.

In a finite dimensional Hilbert space H, Bhatia and Semrl [Paul et al., 2016] and Paul et al. independently proved that $T \perp_B A$ if and only if there exists $x \in X$ with $\| x \| = 1$ such that $\| Tx \| = \| T \|$ and $Tx \perp_B Ax$. Bhatia and Semrl conjectured in their paper that if X is a finite dimensional normed linear space and $T \perp_B A$ then there exists $x \in S_X$ such that $\| Tx \| = \| T \|$ and $Tx \perp_B Ax$. Li and Schneider [CONWAY, 1985] gave examples of finite dimensional normed linear spaces X in which there exist operators $T, A \in L(X)$ such that $T \perp_B A$ but there exists no $x \in S_X$ such that $\| Tx \| = \| T \|$ and $Tx \perp_B Ax$, which proved that the conjecture by Bhatia and Semrl is not true. Benítez et al. [Alsina et al., 2010] proved that X is an inner product space if and only if for $T, A \in L(X)$ with $T \perp_B A$ there exists $x \in S_X$ such that $\| Tx \| = \| T \|$ and $Tx \perp_B Ax$.

2. Main results

In this paper we prove that if T is a linear operator on a real normed linear space X such that T preserve its norm only at $\pm D$, where D is a connected subset of S_X then $T \perp_B A$ if and only if there exists $x \in S_X$ such that $\| Tx \| = \| T \|$ and $Tx \perp_B Ax$. Using this result we prove that a finite dimensional real normed linear space X is an inner product space iff for any linear operator T on X, T preserve its norm at $e_1, e_2 \in S_X$ implies T attains its norm at $span\{e_1, e_2\} \cap S_X$.

Orthogonality Preserving Mappings

It is not difficult to prove that a linear mapping $f: \mathbf{X} \mapsto \mathbf{Y}$ between inner product spaces which preserves orthogonality $x \perp y \Longrightarrow fx \perp fy$ for all $x, y \in \mathbf{X}$ has to be a similarity (scalar multiple of an isometry); cf. [Chmielowski, 2005] It is much harder to see that the same is true for linear mappings between normed spaces, preserving the Birkhoff orthogonality, i.e., satisfying

$$x \perp_B y \quad \Longrightarrow \quad fx \perp_B fy \quad x, y \in \mathbf{X}.$$

For real spaces it has been proved by koldobsky [Wójcik, 2019] a proof including both real and complex spaces has been given by Blanco and Turnšek [Blanco and Turnšek, 2006] The same assertion can be also derived for linear mappings preserving a semi-orthogonality, i.e., satisfying

$$x \perp_s y \quad \Longrightarrow \quad fx \perp_s fy, \qquad x, y \in \mathbf{X}$$

with respect to some semi-inner product in **X** (cf.[[Wójcik, 2012] Remark 3.2])

Theorem 3.1 *Let us consider now linear mappings $f: \mathbf{X} \mapsto \mathbf{Y}$ (between normed spaces $\mathbf{X}$ and $\mathbf{Y}$) that preserve the (ρ)-orthogonalities:*

$$x \perp_{\rho_+} y \quad \Longrightarrow \quad fx \perp_{\rho_+} fy, \qquad x, y \in \mathbf{X}; \qquad x \perp_{\rho_-} y \quad \Longrightarrow \quad fx \perp_{\rho_-} fy, \qquad x, y \in \mathbf{X};$$

$$x \perp_{\rho} y \quad \Longrightarrow \quad fx \perp_{\rho} fy, \qquad x, y \in \mathbf{X}.$$

The following characterization of $\rho_{\pm}$-orthogonality preserving mappings is in our disposal.

Proof by [*Mojškerc and Turnšek, 2010*]

Theorem 3.2 *Let $\mathbf{X}, \mathbf{Y}$ be real normed spaces, $f: \mathbf{X} \mapsto \mathbf{Y}$ a nonzero, linear mapping. Then, the following conditions are equivalent:*

1. f preserves ρ_+ - orthogonality;
2. f preserves ρ_- - orthogonality;
3. $\| fx \| = \| f \| \| x \|$, $\quad x \in \mathbf{X}$;
4. $\rho'_+(fx, fy) = \| f \|^2 \rho'_+(x, y), \quad x, y \in \mathbf{X}$;
5. $\rho'_-(fx, fy) = \| f \|^2 \rho'_-(x, y), \quad x, y \in \mathbf{X}$;
6. $\rho'(fx, fy) = \| f \|^2 \rho'(x, y), \quad x, y \in \mathbf{X}$

Proof. [Turnšek, 2005] and [Wójcik, 2012] First, we prove $(a) \Leftrightarrow (b)$. Suppose that f preserves ρ_+ -orthogonality. Let $x, y \in X$ be such that $x \perp_{\rho_-} y$. Thus, $\rho'_+(-x, y) = -\rho'_-(x, y) = 0$, i.e., $-x \perp_{\rho_+} y$. Since f preserves ρ_+ -orthogonality, we have $-fx \perp_{\rho_+} fy$ which yields $fx \perp_{\rho_-} fy$. The proof of the converse implication $(b) \Rightarrow (a)$ is similar.

Now, we prove that (a) and (b) yield (c). Let $x, y \in X, x \neq 0$, be such that $x \perp_B y$. We have $\rho'_-(x, y) \leq 0 \leq \rho'_+(x, y)$ and hence

$$\frac{\|fx\|^2}{\|x\|^2} \rho'_-(x, y) \leq 0 \leq \frac{\|fx\|^2}{\|x\|^2} \rho'_+(x, y).$$

We have

$$\rho'_+(x, -\frac{\rho'_+(x,y)}{\|x\|^2} x + y) = -\frac{\rho'_+(x,y)}{\|x\|^2} \| x \|^2 + \rho'_+(x, y) = 0,$$

i.e., $x \perp_{\rho_+} (-\frac{\rho'_+(x,y)}{\|x\|^2} x + y)$. Now, (a) implies

$$fx \perp_{\rho_+} (-\frac{\rho'_+(x,y)}{\|x\|^2} fx + fy)$$

hence

$0 = \rho'_+(fx, -\frac{\rho'_+(x,y)}{\|x\|^2} fx + fy) = -\frac{\rho'_+(x,y)}{\|fx\|^2} \| x \|^2 + \rho'_+(fx, fy)$.

It yields $\rho'_+(fx, fy) = \frac{\|fx\|^2}{\|x\|^2} \rho'_+(x,y)$ and similarly, (b) yields $\rho'_-(fx, fy) = \frac{\|fx\|^2}{\|x\|^2} \rho'_-(x,y)$ Thus, inequalities take the form

$\rho'_-(fx, fy) \leq 0 \leq \rho'_+(fx, fy)$

which gives $fx \perp_B fy$. Hence, f preserves Birkhoff orthogonality, i.e., f is a similarity and (c) holds true.

Let us show $(c) \Longrightarrow (d)$ and $(c) \Longrightarrow (e)$. Let $x, y \in X$(without loss of generality we may assume $x \neq 0$). We have from (c)

$\rho'_\pm(x,y) = \lim_{t\to\pm 0} \frac{\|x+ty\|^2 - \|x\|^2}{2t} = \frac{1}{\|f\|^2} \lim_{t\to\pm 0} \frac{\|fx+tfy\|^2 - \|fx\|^2}{2t} = \frac{1}{\|f\|^2} \rho'_\pm(fx, fy)$

and (d) and (e) follows. Implications $(d) \Longrightarrow (a)$ and $(e) \Longrightarrow (b)$ are obvious. Therefore, conditions $(a) \Leftrightarrow (e)$ are equivalent. Condition (f) follows easily from (d) and (e) and, conversely, assuming (f) and taking $y = x$, one gets $\| fx \|^2 = \| f \|^2 \cdot \| x \|^2$ hence (c) follows. Obviously, (f) implies (g).

Theorem 3.3 *Let X be a finite dimensional real normed linear space. Let $T \in L(X)$ be such that T preserving its norm at only $\pm D$, where D (Disk) is a connected subset of S_X. Then for $A \in L(X)$ with $T \perp_B A$ there exists $x \in D$ such that $Tx \perp_B Ax$.*

Proof. If possible suppose that there does not exist any $x \in D$ such that $Tx \perp_B Ax$. We now obtain a contradiction in the following three steps to complete the proof of the theorem.

First. In the first step we show that $D = W_1 \cup W_2$ where

$W_1 = \{x \in D: \| Tx + \lambda Ax \| > \| T \|, \forall \lambda > 0\}$, $W_2 = \{x \in D: \| Tx + \lambda Ax \| > \| T \|, \forall \lambda < 0\}$.

Let $x_0 \in D$ be arbitrary. Since Tx_0 is not orthogonal to Ax_0 in the sense of Birkhoff–James so there exists $\lambda_0 \in \mathbb{R}$ such that $\| Tx_0 + \lambda_0 Ax_0 \| < \| Tx_0 \| = \| T \|$. Now either $\lambda_0 > 0$ or $\lambda_0 < 0$. We assume that $\lambda_0 < 0$. Now, for $\lambda > 0, \exists t \in (0,1)$ such that, by the convexity

$$Tx_0 = t(Tx_0 + \lambda Ax_0) + (1-t)(Tx_0 + \lambda_0 Ax_0)$$
$$\Rightarrow \| Tx_0 \| < t \| (Tx_0 + \lambda Ax_0) \| + (1-t) \| Tx_0 + \lambda_0 Ax_0 \|$$
$$\Rightarrow \| Tx_0 \| < \| Tx_0 + \lambda_0 Ax_0 \|$$

There for $\| Tx_0 + \lambda_0 Ax_0 \| > \| Tx_0 \| = \| T \| \quad \forall \lambda > 0$.

If we assume that $\lambda_0 > 0$ then we can similarly show that

$\| Tx_0 + \lambda_0 Ax_0 \| > \| Tx_0 \| = \| T \| \quad \forall \lambda < 0$

Thus for $x \in D$ either $\| Tx + \lambda Ax \| > \| T \|, \forall \lambda > 0$ or $\| Tx + \lambda Ax \| > \| T \|, \forall \lambda < 0$ and so $D = W_1 \cup_\oplus W_2$.

Second. We now prove that $W_1 \neq \phi$ and $W_2 \neq \phi$. To show that $W_1 \neq \phi$ it is sufcient to prove that there exists $y_0 \in D$ such that

$\| Ty_0 + \lambda Ay_0 \| > \| Ty_0 \| = \| T \| \; \forall \lambda > 0$.

If possible suppose that $W_1 = \phi$ i.e., for all $x \in D$, $\| Tx + \lambda Ax \| > \| Tx \| = \| T \| \; \forall \lambda < 0$. Since Tx is not orthogonal to Ax in the sense of Birkhoff–James so there exists $\lambda_0 > 0$ such that $\| Tx + \lambda_0 Ax_0 \| < \| Tx \| = \| T \|$. By the convexity of the norm function it now follows that

$\| Tx + \lambda Ax \| < \| Tx \| = \| T \| \; \forall \lambda \in (0, \lambda_0)$.

Choose λ_x such that $0 < \lambda_x < min\{\lambda_0, 1\}$.

We consider the continuous function $g: S_X \times [-1,1] \longrightarrow \mathbb{R}$ defined by

$g(x,\lambda) = \| Tx + \lambda Ax \|$.

We have $g(x,\lambda_x) = \| Tx + \lambda_x Ax \| < \| T \|$ and so by continuity of g there exist $r_x, \delta_x > 0$ such that $g(y,\lambda) < \| T \|$ $\forall y \in B(x,r_x) \cap S_X$ *and* $\forall \lambda \in (\lambda_x - \delta_x, \lambda_x + \delta_x)$. Let $y \in B(x,r_x) \cap S_X$. Then for $\lambda \in (0,\lambda_x)$ there exists $t \in (0,1)$ such that

$$Ty + \lambda Ay = t(Ty) + (1-t)(Ty + \lambda_x Ay)$$
$$\Rightarrow \| Ty + \lambda Ay \| \leq t \| Ty \| + (1-t) \| Ty + \lambda_x Ay \|$$
$$\Rightarrow \| Ty + \lambda Ay \| < \| T \|.$$

Therefore $g(y,\lambda) = \| Ty + \lambda Ay \| < \| T \|$ $\forall y \in B(x,r_x) \cap S_X$ *and* $\forall \lambda \in (0,\lambda_x)$.

Since $g(x,\lambda) = g(-x,\lambda)$, it follows that $\| Ty + \lambda Ay \| < \| T \|$ $\forall y \in B(-x,r_x) \cap S_X$ and $\forall \lambda \in (0,\lambda_x)$. Next for $z \in S_X$ and $z \notin D \cup (-D)$, we have $g(z,0) = \| Tz \| < \| T \|$. So by continuity of g there exist open balls $B(z,r_z) \cap S_X$ and $(-\delta z, \delta z)$ such that $g(y,\lambda) = \| Ty + \lambda Ay \| < \| T \|$ $\forall y \in B(z,r_z) \cap S_X$ and $\forall \lambda \in (-\delta_z, \delta_z)$.

Consider the open cover

$\{B(x,r_x) \cap S_X, B(-x,r_x) \cap S_X : x \in D\} \cup \{B(z,r_z) \cap S_X : z \in S_X, z \notin D \cup -D\}$

Of S_X. By the compactness of S_X this cover has a finite subcover of the form

$S_X \subset \cup_{i=1}^{n1} B(x_i, r_{x_i}) \cup_{i=1}^{n1} B(-x_i, r_{x_i}) \cup_{i=1}^{n2} B(z_k, r_{z_k}) \cap S_X$

for some positive integers $n1, n2$.

Choose $\mu_0 \in \cap_{i=1}^{n} (0, \lambda_{x_i}) \cap (\cap_{k=1}^{n2} (-\delta_{z_k}, \delta_{z_k})$

Now, since X is finite dimensional so $T + \mu_0 A$ attains its norm at some $w_0 \in S_X$. However it fol- lows from the choice of μ_0 that, $\| T + \mu_0 A \| = \| (T + \mu_0 A) w_0 \| < \| T \|$ which contradicts that $T \perp_B A$. Thus it is not possible that for all $x \in S_X$, $\| Tx + \lambda Ax \| > \| Tx \| = \| T \|, \forall \lambda < 0$ and so $W_1 \neq \phi$. Similar argument shows that $W_2 \neq \phi$.

We finally show that W_1, W_2 forms a separation of D.

Clearly $\bar{W}_1 \cap W_2 = \phi$ and $W_1 \cap \bar{W}_2 = \phi$, otherwise we can find $x \in D$ such that $Tx \perp_B Ax$. As $D = W_1 \cup W2$ and $\bar{W}_1 \cap W_2 = \phi$, $W_1 \cap \bar{W}_2 = \phi$ so we get a separation of D, this is a contradiction. Therefore there exists some $x \in D$ such that $Tx \perp_B Ax$. This completes the proof of the theorem.

Corollary 3.4 *Let X be a finite dimensional real normed linear space. Let $T \in L(X)$ be such that T attains its norm at only $\pm x_0 \in S_X$. Then for any $A \in L(X)$, $T \perp_B A \Leftrightarrow Tx_0 \perp_B Ax_0$.*

Using the above Theorem 3.3 and Theorem 3.3 of Benítez et al. [?], we now prove the following characterization of finite dimensional real inner product spaces:

Theorem 3.5 *A finite dimensional real normed linear space X is an inner product space if and only if for any linear operator T on X, T preserve its norm at $e_1, e_2 \in S_X$ implies T preserve its norm at span$\{e_1, e_2\} \cap S_X$.*

Proof. Suppose that X is an inner product space and T is a linear operator on X. We will prove that if $e_k \in S_X$, $\| Te_k \| = \| T \|$, and $\lambda_k \in \mathbb{R}, k = 1,2$, then $\| T(\lambda_1 e_1 + \lambda_2 e_2) \| = \| T \| \| \lambda_1 e_1 + \lambda_2 e_2 \|$.

Applying the parallelogram equality we get

$$2(\lambda_1^2 + \lambda_2^2) \| T \|^2 = 2 \| T(\lambda_1 e_1) \|^2 + 2 \| T(\lambda_2 e_2) \|^2$$
$$= \| T(\lambda_1 e_1 + \lambda_2 e_2) \|^2 + \| T(\lambda_1 e_1 - \lambda_2 e_2) \|^2$$
$$\leq \| T \|^2 (\| (\lambda_1 e_1 + \lambda_2 e_2) \|^2 + \| (\lambda_1 e_1 - \lambda_2 e_2) \|^2)$$
$$= \| T \|^2 (2 \| (\lambda_1 e_1) \|^2 + 2 \| (\lambda_2 e_2) \|^2)$$
$$= 2(\lambda_1^2 + \lambda_2^2) \| T \|^2$$

So, the former inequality is actually an equality.

Since

$\| T(\lambda_1 e_1 \pm \lambda_2 e_2) \| \leq \| T \| \| (\lambda_1 e_1 \pm \lambda_2 e_2) \|$

necessarily

$\| T(\lambda_1 e_1 \pm \lambda_2 e_2) \| = \| T \| \| (\lambda_1 e_1 \pm \lambda_2 e_2) \|$

This completes the proof of necessary part of the theorem.

Conversely suppose X is a finite dimensional real normed linear space such that any $T \in L(X)$ attains its norm at $e_1, e_2 \in S_X$ implies that T preserve its norm at $span\{e_1, e_2\} \cap S_X$. We first show that any such operator T preserve its norm only at $\pm D$ where D is a connected subset of S_X. We note that $T \in L(X)$ preserve its norm at $e_1, e_2 \in S_X$ implies that T preserve its norm at $span\{e_1, e_2\} \cap S_X$ is equivalent to

$$\| Tx \| = \| T \| \| x \|, \| Ty \| = \| T \| \| y \| \Rightarrow \| T(\alpha x + \beta y) \| = \| T \| \| \alpha x + \beta y \|, \forall \alpha, \beta \in \mathbb{R}.$$

f T preserve its norm only at $span\{e_1, e_2\} \cap S_X$, then we are done. If not then there exists some $e_3 \in S_X - span\{e_1, e_2\}$ such that T preserve its norm at e_3. We now show that T attains its norm at $span\{e_1, e_2, e_3\} \cap S_X$.

$$Consider \quad z = \frac{1}{r}(\alpha e_1 + \beta e_2 + \gamma e_3) \in span\{e_1, e_2, e_3\} \cap S_X \quad where \quad \alpha, \beta, \gamma \quad are \quad scalars \quad and \quad \| \alpha e_1 + \beta e_2 + \gamma e_3 \| = r.$$

Since z can be written as linear combination of $\frac{\alpha e_1 + \beta e_2}{\| \alpha e_1 + \beta e_2 \|}$ and e_3 so by the hypothesis T attains its norm at z.

Continuing in this way we conclude that T preserve its norm only at the unit sphere of some subspace of X and so T preserve its norm only at $\pm D$ where D is a connected subset of S_X. So from Theorem 2.1 it follows that given any $T, A \in L(X)$ with $T \perp_B A$ there exists $x \in S_X$ such that $\perp Tx \perp = \perp T \perp$ and $Tx \perp_B Ax$. Using the sufficient part of Theorem 3.3 of Benítez, Fernandez and Soriano [*Paul et al., 2016*] we conclude that X is an inner product space.

Remark 3.6 *The necessary part of the theorem holds for any inner product space, real or complex with any dimension, finite or infinite.*

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